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The Liouville condition and Nambu mechanics

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Abstract. It is shown that if a system of coupled differential equations satisfies the Louville condition it is not necessarily constructed according to the Nambu prescripton. The relation of the number of time-independent integrals of the system to the required number of Hamiltonians is explored. The extended version of Nambu mechanics that admits singlets is related to the generalized Hamiltonian version of dynamics. All features are explored within one specific example in three-dimensional phase space.

1. Introduction

In the Nambu version of dynamics [1] the phase space is spanned by the dynamical variables x_i, $i = 1, \ldots, n$ (n even or odd); this is called an n-dimensional multiplet. The particular form of the time evolution equations for these variables implies that the Liouville condition, $\partial[\mathrm{d}x_i/\mathrm{d}t]/\partial x_i = 0$, is identically satisfied. In fact, the Nambu prescription requires that $(n-1)$ functions of the dynamical variables $H_1, \ldots, H_{n-1}$—called the Hamiltonians of the system—be given; then the evolution equation for any function $A = A(x_1, \ldots, x_n)$ is

$$\mathrm{d}A/\mathrm{d}t = \partial(A, H_1, \ldots, H_{n-1})/\partial(x_1, \ldots, x_n) \tag{1}$$

where $\partial(\ldots)/\partial(\ldots)$ is a Jacobian; taking $A = x_i$ in (1), differentiating partially with respect to x_i and summing over i leads to the Liouville condition. A direct consequence of (1) is that $\mathrm{d}H_i/\mathrm{d}t = 0$, $i = 1, \ldots, n-1$.

In this paper the inverse problem is studied, using one particular example: given a system of coupled ordinary differential equations of the form $\mathrm{d}x_i/\mathrm{d}t = F_i$ such that the functions F_i satisfy $\partial F_i/\partial x_i = 0$ it is required to find under what conditions there exist $(n-1)$ functions $H_1, \ldots, H_{n-1}$ that are time-independent so that the given system is generated according to (1). The main result is that only for certain values of the parameters that define the example is this possible. Although three-dimensional phase space is used most of the conclusions are valid for any dimension n.

The conclusion reached does not agree with previous ones which assert that, the Liouville condition is '. . . *sufficient* as well as necessary. . . ' (see [2]) to have (1) and that the use of n integrals—one of which is explicitly time-dependent (see [3])—also leads to (1). It turns out that in the specific illustrations considered in [3] the conclusion is true but this is not so in the general case. The system $\mathrm{d}x_i/\mathrm{d}t = F_i(x_1, \ldots, x_n)$ admits possibilities that do not coincide with (1); among them:

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(i) the separation of the dynamical variables into a singlet coupled to an $(n-1)$-dimensional multiplet (see [4] for a discussion of the specific case $n = 4$);

(ii) the system is Hamiltonian in the generalized sense of [5–7];

(iii) if n is not prime then the dynamical variables can be grouped into a number of multiplets all of the same dimension (see [8])—this case is, however, closely related to the Nambu prescription in the sense that the evolution equation for a dynamical variable is a sum of Jacobians instead of a single Jacobian as in (1) and that if $n = ms$ there are $(s-1)$ Hamiltonians (which are, of course, functions of all the variables that span phase space) when there are m multiplets;

(iv) it does not admit one of the possibilities (i)–(iii); this subfamily will be called non-Hamiltonian.

In the particular example considered in this paper it is found:

(i) that the system is Hamiltonian if it corresponds to a Nambu multiplet—this is Ruggeri's result [9];

(ii) that the triplet and the singlet coupled to a doublet are unconnected; and

(iii) that the singlet coupled to a doublet is not Hamiltonian.

Notation and conventions. $x_i' = \mathrm{d}x_i/\mathrm{d}t$, $\mathrm{D}_i = \partial/\partial x_i$, $\mathrm{d}_i = \partial/\partial y_i$, $\mathrm{D}_t = \partial/\partial t$, $\boldsymbol{x} = (x_i, \ldots, x_n)$, e_{ijk} is the Levi-Civita tensor and the summation convention is used throughout. Boldface letters denote vectors.

2. General remarks on dynamical systems

The dynamical system considered has the form

$$\mathrm{d}x_i/\mathrm{d}t = F_i(\boldsymbol{x}) \qquad i = 1, \ldots, n \tag{2}$$

where the F_i are given functions of the dynamical variables such that $\mathrm{D}_i F_i = 0$. The main purpose of this paper is to investigate under what conditions the right-hand side of system (2) can be generated following the prescription specified in (1). Since in (1) knowledge of the Hamiltonians is crucial and these functions are integral invariants of the system (1) special attention is paid to the integral invariants of system (2). Once the set of integral invariants is known it is also known how many of them—or functions of them (which are also invariants)—do not explicitly involve the time. If this number is $(n-1)$ then the system (2) may be cast in the form (1)—see remark 2.1.

The integral invariants of system (2) satisfy

$$R_i(\boldsymbol{x}, t) = C_i \qquad i = 1, \ldots, n \tag{3}$$

where the C_i's are constants. The set of invariants (3) is also called the integral inequivalent of (2).

Remark 2.1. The particular case of the Nambu mechanics has the first $(n-1)$ R's time-independent—they correspond to the Hamiltonians—and R_n explicitly time-dependent. Moreover, it is required that R_n be a monotonic function of time. To see this recall that Cohen's procedure [3] requires knowledge of $(n-1)$ time-independent integral invariants and a time-dependent one; from them the components of the velocity are computed. The result is that the x_i' have, as a factor, the time derivative of the last integral invariant. If this factor vanishes for a particular value of time, an equilibrium point is generated which always belongs to the solution; the solution reduces, then, to this point.

The set of integrals (3) gives rise to (2) in the following sense: the solution obtained from (2) and the functions $x_i(t)$ extracted from (3) are the same or equivalently, the $\mathrm{d}x_i/\mathrm{d}t$'s obtained from (3) are the same as the ones exhibited in (2). This gives sense to the assertion that (3) is the integral equivalent of (2). Any of the integrals (3) satisfies identically $\mathrm{d}R_i = 0$ after use of the set (2). If the starting point is (3) then the equation satisfied by each R_i is [10]

$$U^s \mathrm{D}_s R_i + \mathrm{D}_t R_i = 0 \qquad i = 1, \ldots, n \tag{4}$$

where the U^s are given by

$$\partial(R_1, \ldots, R_n)/\partial(x_1, \ldots, x_n) U^s = -\partial(R_1, \ldots, R_n)/\partial(x_1, \ldots, x_{s-1}, t, x_{s+1}, \ldots, x_n). \tag{5}$$

Remark 2.2. It is easy to see that if $R_1, \ldots, R_{n-1}$ are time-independent then $U^s = x_s'$ after introducing a new time variable as in [3] and the conditions of remark 2.1 are satisfied.

To conclude with these very general remarks it is important to say that all functions R_i are functionally independent so that none of the relations (3) can be obtained as combinations of the others; if this was the case some of the relations in (3) could be dismissed leaving less than n in number; this implies that more than one independent variable appears. The case considered in this paper corresponds to the so-called completely integrable one (for further details see [10]).

As already stated, to approach the problem of generating system (2) according to (1) it is important to determine the minimum number of time-dependent integrals associated to (2). To this end attention is paid to the set of ordinary differential equations equivalent to $\mathrm{d}R_i = 0$

$$F_1^{-1}\mathrm{d}x_1 = F_2^{-1}\mathrm{d}x_2 = \cdots = F_n^{-1}\mathrm{d}x_n = \mathrm{d}t. \tag{6}$$

If the solution to the subsystem of differential equations that does not involve $\mathrm{d}t$ consists of $(n-1)$ functionally independent functions then only one time-dependent integral will appear and the system will be generated according to the Nambu prescription if the monotonicity condition is satisfied (see remark 2.1). But if this number is less than $(n-1)$, say k, then the system has $(n-k)$ time-dependent integrals; there is still the possibility that enough functions can be constructed from these time-dependent integrals that do not involve explicitly the time so as to have $(n-1)$ in all, the system may be of the Nambu type. Otherwise it will be not. These features have to be explored in each particular case. The study of the Nambu mechanics starts from knowledge of the $(n-1)$ Hamiltonians—as given input—required to generate the equations of motion according to (1). The point of view adopted in this paper is that the primary information is (2) and the number of Hamiltonians—if they exist—should be an outcome of the study of this system.

Remark 2.3. It is well known that for a system of first-order differential equations such as (2) a Jacobi multiplier always exists so that it can be cast in the form (1)—this is in fact the role of the Jacobi multiplier—with integrals—that replace the Hamiltonians—which are, in general, functions of $(\boldsymbol{x}, t)$ see ([10, 11]). The requirement that $(n-1)$ of the integrals do not involve the time restricts the possible systems allowed in a particular situation; it is this extra condition that impedes stating that the Liouville condition is equivalent to the Nambu form of dynamics and invalidates some of the conclusions in [2].

Before proceeding the definitions of a singlet coupled to a doublet and of generalized Hamiltonian dynamics are briefly summarized. In three-dimensional phase space the system (2) is a singlet coupled to a doublet [4] if there is a function H that generates it according to $x_1' = D_2 H$, $x_2' = -D_1 H$, $x_3' = D_3 H$. It follows that H satisfies $H' = (D_3 H)^2$ and if the Liouville condition is imposed the result is $D_3^2 H = 0$; the function H is, therefore, not a constant of the motion and must be, at most, linear in x_3. The system (2) is Hamiltonian in the generalized sense ([5–7]) if it is generated from an antisymmetric matrix K that satisfies the Jacobi identity—see (34) below—and a function G according to $x_i' = K_{ij} D_j G$ which implies that G is an integral invariant. The Liouville condition leads to $(D_i K_{ij}) D_j G = 0$; this relation is identically satisfied if K is a constant matrix, otherwise it restricts K and/or the Hamiltonian function G.

From now on the particular case $n = 3$ will be considered. There are three independent integral invariants for the system (2) for which three maximal situations can be envisaged: the set of three integral invariants has either one time-dependent and two time-independent or two time-dependent and one time-independent or three time-dependent functions. A maximal situation will be defined as one for which the set of integrals (3) has the maximum number of time-independent functions in a preferred set of variables. Now the possibilities are clear: if two time-independent functions exist, provided the time-dependent integral is monotonic in time, the system will be related to the Nambu mechanics and the Hamiltonians will be functions of the time-independent integrals; this is Cohen's result [3]—it has to be noted that, in this case, the system is also Hamiltonian in the generalized sense. If there is only one time-independent integral, the system could be Hamiltonian if it can be generated from this integral or by an appropriate function of it according to the prescription of [5–7]. If there is an explicitly time-independent function which is, however, not constant in time but which generates the system according to [4] then it represents a singlet coupled to a doublet; all other cases are non-Hamiltonian. From what has been said it is clear that satisfaction of the Liouville condition does not guarantee that the system of differential equations will be constructed according to the Nambu prescription.

3. A specific example

The features that have been described in the previous section will be illustrated considering a simple case. The example is defined by the third-order equation

$$x''' = F(x, x', x''). \tag{7}$$

Remark 3.1. (7) includes the equation for a charged particle moving under the action of a Newtonian and the classical radiation force [12, 13]. A particular case of $F(x, x', x'')$ is Weber's version of electrodynamics [12–15].

Defining $x_1 = x$, $x_2 = x'$ and $x_3 = x''$ (7) takes the form

$$x_1' = x_2 \qquad x_2' = x_3 \qquad x_3' = F(x_1, x_2, x_3). \tag{8}$$

Imposing now the Liouville condition it is found that the function $F(x_1, x_2, x_3)$ must depend on x_1 and x_2 only (i.e. it is a function of position and velocity). For purposes of illustration $F(x_1, x_2, x_3)$ will be taken as a linear function: $F(x_1, x_2, x_3) = ax_1 + bx_2$, with a and b constants.

The integrals T of

$$x_1' = x_2 \qquad x_2' = x_3 \qquad x_3' = ax_1 + bx_2 \tag{9}$$

are defined by

$$x_i' D_i T + D_t T = 0 \tag{10}$$

with equivalent ordinary differential equations

$$x_2^{-1} dx_1 = x_3^{-1} dx_2 = (ax_1 + bx_2)^{-1} dx_3 = dt. \tag{11}$$

The solution of (11) is easily found: constants r, s_1, s_2 and s_3 exist such that

$$\begin{aligned}(s_1\,\mathrm{d}x_1 + s_2\,\mathrm{d}x_2 + s_3\,\mathrm{d}x_3)/(s_1x_2 + s_2x_3 + s_3[ax_1 + bx_2]) &= \mathrm{d}t \\ &= (s_1\,\mathrm{d}x_1 + s_2\,\mathrm{d}x_2 + s_3\,\mathrm{d}x_3)/[r(s_1x_1 + s_2x_2 + s_3x_3)]\end{aligned} \tag{12}$$

this leads to the system

$$rs_1 = s_3a \qquad rs_2 = s_1 + s_3b \qquad rs_3 = s_2 \tag{13}$$

which has a non-trivial solution for s_1, s_2 and s_3 if r satisfies

$$r^3 - br - a = 0. \tag{13a}$$

Once the roots have been computed, there are three sets of values (s_1, s_2, s_3) and from (11) the integral invariants are

$$L_i(x_1, x_2, x_3) = P_i(t)\exp(r_it) \qquad i = 1, 2, 3 \tag{14}$$

where for each i a different set (s_1, s_2, s_3) has to be taken and $P_i(t)$ is, at most, a first-order polynomial. If there are three different values for r then the P_i's are constant and from (14) there are two time-independent integrals

$$L_1^{1/r_1}(x_1, x_2, x_3) = L_2^{1/r_2}(x_1, x_2, x_3) \tag{15}$$

$$L_2^{1/r_2}(x_1, x_2, x_3) = L_3^{1/r_3}(x_1, x_2, x_3) \tag{16}$$

and the third integral invariant is explicitly time-dependent—it can be taken as any of the equations (14). In this case the time dependence is monotonic and, according to Cohen's procedure [3], the system can be brought into the Nambu standard form. When there is a double root a single integral invariant that is time-independent can be constructed; this is due to the fact that in (14) $r_2 = r_3$ (say) and then the right-hand side for $i = 3$ is of the form $t\exp(r_2t)$ whose time derivative vanishes for $t = -1/r_2$. In this case any attempt to use Cohen's procedure obviously fails first because the function is not monotonic in time and second because there is only one integral invariant that is time-independent. The case of three equal roots is trivial since this requires $a = b = 0$.

Remark 3.2. In [16] the motion of a radiating charged particle in an external electric field was considered. The explicit form of the set of three equations is different from the ones presented here. The Liouville condition is not fulfilled; the main aim in [16] is to cast the system in generalized Hamiltonian form (see also [9]).

3.1. A Nambu triplet

Up to this stage it has been shown that a system of coupled ordinary differential equations that satisfies the Liouville condition does not have two time-independent integral invariants in general. The results obtained depend on the discriminant of (13a) but this in turn reflects on the particular values of the parameters a and b. Define $B = (a/2)^2 - (b/3)^3$ then the case of interest corresponds to $B < 0$. The roots of the cubic equation are all real and the two integral invariants are given by (15) and (16) with

$$L_i = u_i[ax_1/r_i + r_ix_2 + x_3] \qquad i = 1, 2, 3. \tag{17}$$

It is now an easy matter to see that only for $a = 0$ do the relations (15) and (16) generate—according to the Nambu prescription—the system (9). The explicit form of the Hamiltonian is

$$H_1 = x_1 - x_3/b \qquad 2H_2 = bx_2^2 - x_3^2. \tag{18}$$

Proof. Write H_1 as a linear and H_2 as a quadratic function of $\boldsymbol{x}$, then replace in (1) and require that after equating with (9) the result be identically satisfied. This leads to a set of nine equations that are consistent only if $a = 0$. The expressions for H_1 and H_2 are given in (18). □

Alternative proof. Consider the two time-independent integral invariants as in (15) and (16); define the two Hamiltonians as any function of these combinations of L's. Use them in the Nambu prescription so as to reproduce (9). The results involve the functions L_1 in such a way that to recover a linear function one of the roots of the cubic equation must vanish; this leads to a linear and a quadratic Hamiltonian. A zero root means $a = 0$.

Remark 3.3. If two of the roots satisfy $r_2 = r_3^*$ then in (14) the invariants for $i = 2$ and 3 are replaced by their real and imaginary parts; with these functions and the one for $i = 1$ only one function not involving the time can be constructed.

3.2. A singlet and a doublet

In this case it is necessary to construct a single function H which does not depend explicitly on time but which is also not a constant of the motion. The integral invariants already constructed are of no use in this case. The equations defining H are

$$x_1' = \mathrm{D}_2 H = x_2 \qquad x_2' = -\mathrm{D}_1 H = x_3 \qquad x_3' = \mathrm{D}_3 H = ax_1 + bx_2 \tag{19}$$

which has a solution only for $a = -1$, $b = 0$ given by

$$H = x_2^2/2 - x_1 x_3. \tag{20}$$

This result shows that the triplet and singlet are clearly separated by the numerical values of the constants. There is, however, the possibility that (9) with $a = 0$ can be transformed into

$$y_1' = y_2 \qquad y_2' = y_3 \qquad y_3' = -y_1 \tag{21}$$

which is (9) with $a = -1$, $b = 0$. Write both systems as

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & b & 0 \end{pmatrix} \boldsymbol{x} = M\boldsymbol{x} \tag{22}$$

$$\boldsymbol{y}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \boldsymbol{y} = N\boldsymbol{y} \tag{23}$$

and define the transformations $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y})$, $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x})$ which are assumed to be one inverse of the other. Define $[i, j] = \mathrm{d}_j f_i$ evaluated at $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x})$; then the following relations are obtained

$$x_i' = \mathrm{d}_j f_i y_j' = \mathrm{d}_j f_i N_{jk} y_k = M_{ij} x_j \tag{24}$$

using the explicit forms of M and N and $y_k = g_k(\boldsymbol{x})$ it follows that

$$x_2 = [1, 1]g_2 + [1, 2]g_3 - [1, 3]g_1 \tag{25}$$

$$x_3 = [2, 1]g_2 + [2, 2]g_3 - [2, 3]g_1 \tag{26}$$

$$bx_2 = [3, 1]g_2 + [3, 2]g_3 - [3, 3]g_1 \tag{27}$$

from (25) and (27) g_1 can be expressed as function of g_2 and g_3 after using the functional independence of the f_i's. This result implies that the g_j's are not functionally independent which violates the initial assumption. Therefore the two systems cannot be transformed into one another.

Remark 3.4. The result just found is not general. A non-Nambu system can, in particular cases, be connected to a Nambu system; see [17] for an example.

3.3. A Hamiltonian system

In this case the system (9) should be generated from a single Hamiltonian function $G(\boldsymbol{x})$ according to

$$x'_j = K_{jn}\mathrm{D}_n G \tag{28}$$

where K is an antisymmetric matrix that satisfies the Jacobi identity. A Hamiltonian $G(\boldsymbol{x})$ quadratic in the coordinates will be assumed, in this case K is a matrix with constant entries so that the Jacobi identity is trivial; write

$$2G(\boldsymbol{x}) = z_{ij}x_i x_j \tag{29}$$

where the z_{ij} are elements of a symmetric matrix Z; using (28) leads to the equations

$$x'_j = (KZ)_{jn}x_n \tag{30}$$

which, after use of (9) and requiring that these equations be satisfied identically, imply nine equations for the nine unknown quantities (z_{11}, z_{12}, z_{13}, z_{22}, z_{23}, z_{33}; K_{12}, K_{13}, K_{23}). It is found that these equations are consistent only if $a = 0$, that is, in the case of a Nambu triplet. This is Ruggeri's result on the formulation of Nambu mechanics as a singular generalized Hamiltonian mechanics [9]. The same result is obtained if the Hamiltonian is chosen as a linear function with the matrix elements of K linear in the coordinates.

In the general case the equations that have to be solved are

$$x'_1 = K_{12}\mathrm{D}_2 G + K_{13}\mathrm{D}_3 G = x_2 \tag{31}$$

$$x'_2 = -K_{12}\mathrm{D}_1 G + K_{23}\mathrm{D}_3 G = x_3 \tag{32}$$

$$x'_3 = -K_{13}\mathrm{D}_1 G - K_{23}\mathrm{D}_2 G = ax_1 + bx_2 \tag{33}$$

and the Jacobi identity

$$K_{rs}\mathrm{D}_s K_{tu} + K_{ts}\mathrm{D}_s K_{ur} + K_{us}\mathrm{D}_s K_{rt} = 0 \tag{34}$$

where both $G(\boldsymbol{x})$ and the matrix $K(\boldsymbol{x})$ have to be determined. Define the vector $\boldsymbol{h} = (K_{23}, -K_{13}, K_{12})$, then the Jacobi identity reduces to

$$\boldsymbol{h} \cdot \mathrm{curl}\, \boldsymbol{h} = 0 \tag{35}$$

whose general solution is $\boldsymbol{h} = f\,\mathrm{grad}\, g$ with f and g arbitrary functions. Replacing this result in (28) leads, after use of $K_{ij} = e_{ijk}h_k$, to

$$x'_i = f e_{ijk}\mathrm{D}_k g \mathrm{D}_j G \tag{36}$$

which reduces to (1) if $f = 1$ or, if f does not vanish, after a change to a new time parameter w such that $\mathrm{D}_t w = f$. Other cases lead to systems that are different from a Nambu triplet but that are Hamiltonian in the generalized sense. If the Liouville condition is used in (36) it follows

$$e_{ijk}\mathrm{D}_i f \mathrm{D}_j g \mathrm{D}_k G = 0 \tag{37}$$

which includes more cases than in the Nambu mechanics. As a result the generalized Hamiltonian dynamical scheme is wider than the Nambu one; what is not clear is the relation between the usual Hamiltonian scheme and the Nambu scheme; a partial answer to this question has been given in [18].

It is easy to show that in this example the system is Hamiltonian in the generalized sense. In fact, write the Hamiltonian H as (q_1 and q_2 constants)

$$2H = q_1 x_1^2 + 2q_1 x_1 x_3 + q_2 x_2^2 + (q_1 + q_2)x_3^2 \tag{38}$$

and the matrix elements of K as $K_{12} = K_{23} = 1/q_2$, $K_{13} = 0$.

4. Conclusions

It has been shown that if the Liouville condition is satisfied then the system of coupled differential equations is not necessarily derived from the Nambu prescription. A specific example—see (9)—has been used to illustrate the main results; this example turned out to be sufficiently simple and at the same time rich enough to exhibit, to a large extent, the main properties. It has been found that if $a \neq 0$ then the system is not Nambu; if $a = 0$ and b arbitrary the system is Nambu while if $b = 0$ and $a = -1$ the system is a singlet coupled to a doublet. The system is Hamiltonian in the generalized sense in a wide variety of situations which include the triplet; the cases in which the Hamiltonian case is not equivalent to a triplet have been explicitly exhibited. Of course there is a range of values of the parameters that do not fall into any of the above categories so that there is still enough room for other possibilities not considered here. What is important is that the role played by the Liouville condition in relation to the Nambu prescription has been clarified and the main moral is that any specific case has to be studied individually.

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